

Graphical Calculus on Representations of Quantum Lie Algebras

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Abstract

The main theme of this thesis is the representation theory of quantum Lie algebras. We develop graphical calculation methods. Jones-Wenzl projectors for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ are very powerful tools to find not only invariants of links but also invariants of 3-manifolds. We find single clasp expansions of generalized Jones-Wenzl projectors for simple Lie algebras of rank 2. Trihedron coefficients of the representation theory for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ has significant meaning and it is called $3j$ symbols. Using single clasp expansions for $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$, we find some trihedron coefficients of the representation theory of $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$. We study representation theory for $\mathcal{U}_q(\mathfrak{sl}(4, \mathbb{C}))$. We conjecture a complete set of relations for $\mathcal{U}_q(\mathfrak{sl}(4, \mathbb{C}))$.

Chapter 1

Introduction

There has been big progress in the theory bridging Lie algebras and low-dimensional topology. These developments are based on quantum groups, braided categories and new invariants of knots, links and 3-manifolds. After the discovery of the Jones polynomial [Jon85] [Jon87], Reshetikhin and Turaev [RT90] [RT91] showed that braided categories derived from quantum groups provide a natural generalization of the Jones polynomial.

One of the developments is that a category of tangles with skein relations leads to a braided category. If we decorate each component of a tangle by a module over a simple Lie algebra, the category becomes a ribbon category. Then we can get an invariant of links, and sometimes 3-manifolds, from a functor constructed in [Tur94]. To develop this theory further, we would like to generalize the Jones-Wenzl projectors in the Temperley-Lieb algebra to the quantization of other simple Lie algebras. The n -th Temperley-Lieb algebra is realized as the algebra of intertwining operators of the $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ -module $V_1^{\otimes n}$, where V_1 is the two-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$. For each n , the algebra T_n has an idempotent f_n such that $f_n x = x f_n = \epsilon(x) f_n$ for all $x \in T_n$ and $f_n f_n = f_n$, where ϵ is an augmentation. These idempotents were first discovered by V. Jones [Jon83] and H. Wenzl [Wen87], and

they found a recursive formula:

$$f_n = f_{n-1} + \frac{[n-1]}{[n]} f_{n-1} e_{n-1} f_{n-1}.$$

So they are named *Jones-Wenzl idempotents (Projectors)*. Kuperberg [Kup96] defines a generalization of the Temperley-Lieb category to the three rank two Lie algebras $\mathfrak{sl}(3, \mathbb{C})$, $\mathfrak{sp}(4)$ and G_2 . These generalizations are called combinatorial rank two spiders. Also he has proved that Jones-Wenzl projectors exist for simple Lie algebras of rank 2 and he called them *clasps*. We will study how they can be expanded inductively in Chapter 2.

The skein module theory allows not only links but also graphs. The invariants of the two simplest nontrivial trivalent graphs, the trihedron and tetrahedron, have significant meaning and they are called $3j$ and $6j$ symbols. So we can naturally ask how to compute trihedron coefficients for $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$ as suggested in [Kup96]. In Chapter 3, we will apply our clasp expansions to find some trihedron coefficients for $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$.

Kuperberg's generalization of the Temperley-Lieb algebra is a set of generators and relations for each rank 2 Lie algebra [Kup96]. The generators are easy to find, but to get a complete set of relations is a challenging problem. In Chapter 4, we follow Kuperberg's method to find some relations. We conjecture a complete set of relations of $\mathcal{U}_q(\mathfrak{sl}(4, \mathbb{C}))$.

1.1 Preliminaries

For simple terms, we refer to [Hum72] [Kas95] and [CKT97].

Quantum integers are defined as

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$$

$$[0] = 1$$

$$[n]! = [n][n-1] \dots [2][1]$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

Let $\mathfrak{sl}(n, \mathbb{C})$ be the Lie algebra of complex $n \times n$ -matrices with trace zero. Let $E_{i,j}$ be the elementary matrix whose entries are all zero except 1 in the (i, j) -th entry. Let $E_i = E_{i,i+1}$, $F_i = E_{i+1,i}$ and $H_i = E_{i,i} - E_{i+1,i+1}$ where $1 \leq i \leq n-1$, then they generate $\mathfrak{sl}(n, \mathbb{C})$ with relations:

$$\begin{aligned} [H_i, H_j] &= 0 && \text{for } i, j = 1, 2, \dots, n-1 \\ [H_i, E_j] &= \alpha_j(H_i)E_j && \text{for } 1 \leq i, j \leq n-1 \\ [H_i, F_j] &= -\alpha_j(H_i)F_j && \text{for } 1 \leq i, j \leq n-1 \\ [E_i, F_j] &= \delta_{ij}H_i && \text{for } 1 \leq i, j \leq n-1 \\ [E_i, E_j] &= 0 && \text{if } |i-j| \geq 2 \\ [F_i, F_j] &= 0 && \text{if } |i-j| \geq 2 \\ [E_i, [E_i, E_j]] &= 0 && \text{if } |i-j| = 1 \\ [F_i, [F_i, F_j]] &= 0 && \text{if } |i-j| = 1 \end{aligned}$$

where α_i is a linear form defined by

$$\alpha_j(H_i) = \begin{cases} 2, & \text{if } i = j \\ -1, & \text{if } |i-j| = 1 \\ 0, & \text{Otherwise} \end{cases}$$

The quantum group $\mathcal{U}_q(\mathfrak{sl}(n, \mathbb{C}))$ is an associative algebra over $\mathbb{C}(q)$ with generators, E_i, F_i, K_i^\pm with $1 \leq i \leq n-1$, and relations:

$$\begin{aligned}
K_i K_i^{-1} &= 1 = K_i^{-1} K_i && \text{for } i = 1, 2, \dots, n-1 \\
K_i E_j &= q^{\alpha_j(H_i)} E_j K_i && \text{for } i, j = 1, 2, \dots, n-1 \\
K_i F_j &= q^{-\alpha_j(H_i)} F_j K_i && \text{for } i, j = 1, 2, \dots, n-1 \\
[E_i, F_j] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} && \text{for } i, j = 1, 2, \dots, n-1 \\
[E_i, E_j] &= 0 && \text{if } |i - j| \geq 2 \\
[F_i, F_j] &= 0 && \text{if } |i - j| \geq 2 \\
E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 &= 0 && \text{if } |i - j| = 1 \\
F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 &= 0 && \text{if } |i - j| = 1
\end{aligned}$$

Let \mathfrak{h} be the Lie subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ generated by H_i and let $\Lambda \in \mathfrak{h}$ be the integral lattice of linear forms on H_i where $n-1 \geq i \geq 1$. Let $\lambda \in \Lambda, \epsilon = (\epsilon_1, \dots, \epsilon_{n-1})$, then there is a unique universal highest weight module, a *Verma Module*, with highest weight (λ, ϵ) . $M(\lambda, \epsilon)$ has a unique simple quotient $L(\lambda, \epsilon)$ which is highest module with highest weight (λ, ϵ) . Then $L(\lambda, \epsilon)$ is finite dimensional if and only if λ is dominant weight. One can see that as $\mathcal{U}_q(\mathfrak{sl}(n, \mathbb{C}))$ module

$$L(\lambda, \epsilon) \cong L(\lambda, 0) \otimes L(0, \epsilon).$$

So we can study $L(\lambda, 0)$ which is denoted by $L(\lambda)$. Then there is a theorem which connects studies of $\mathcal{U}_q(\mathfrak{sl}(n, \mathbb{C}))$ modules and $\mathfrak{sl}(n, \mathbb{C})$ modules.

Theorem 1.1 [CKT97]

i) Any finite dimensional simple $\mathcal{U}_q(\mathfrak{sl}(n, \mathbb{C}))$ module is of the form $L(\lambda, \epsilon)$ where λ is dominant weight and $\epsilon \in (\mathbb{Z}/2\mathbb{Z})^{n-1}$.

ii) The character $ch(L(\lambda))$ is given by the same formula as the character of simple $\mathfrak{sl}(n)$ module parameterized by the same highest weight.

iii) The multiplicity of a simple module $L(\nu)$ in the decomposition of the tensor product $L(\lambda) \otimes L(\mu)$ of two simple modules is the same as for the decomposition of the corresponding $\mathfrak{sl}(n, \mathbb{C})$ module.

Chapter 2

Single Clasp Expansions for Rank 2 Lie Algebras

2.1 Introduction

Let T_n be the n -th Temperley-Lieb algebra with generators, $1, e_1, e_2, \dots, e_{n-1}$, and relations:

$$\begin{aligned} e_i^2 &= -(q^{\frac{1}{2}} + q^{-\frac{1}{2}})e_i \\ e_i e_j &= e_j e_i \quad \text{if } |i - j| \geq 2 \\ e_i &= e_i e_{i \pm 1} e_i \end{aligned}$$

For each n , the algebra T_n has an idempotent f_n such that $f_n x = x f_n = \epsilon(x) f_n$ for all $x \in T_n$, where ϵ is an augmentation. These idempotents were first discovered by V. Jones [Jon83] and H. Wenzl [Wen87]. They found a recursive formula:

$$f_n = f_{n-1} + \frac{[n-1]}{[n]} f_{n-1} e_{n-1} f_{n-1}$$

as in the following figure where we use a red box to represent f_n :

$$\begin{array}{c} n \\ \text{---} \\ n \end{array} = \begin{array}{c} n-1 \\ \text{---} \\ n-1 \end{array} \left| \begin{array}{c} n-1 \\ \text{---} \\ n-1 \end{array} \right| + \frac{[n-1]}{[n]} \begin{array}{c} n-1 \\ \text{---} \\ n-2 \\ \text{---} \\ n-1 \end{array} \quad (2.1)$$

So they are named *Jones-Wenzl idempotents(projectors)*. We will recall an algebraic definition of Jones-Wenzl projectors in section 1. We refer to [Kup96] for definitions, notation and simple calculations. We provide single clasp expansions of generalized Jones-Wenzl projectors for $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$ in section 2. In section 3 we study single clasp expansions of generalized Jones-Wenzl projectors for $\mathcal{U}_q(\mathfrak{sp}(4))$.

2.2 Single Clasp Expansion for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$

First we recall another definition of Jones-Wenzl projectors and single clasp expansions for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$. Then we use it to find trihedral coefficients for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$.

2.2.1 Jones-Wenzl Projector for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$

Let us give a precise definition [Kho97] of a *clasp* for $\mathcal{U}(\mathfrak{sl}(2, \mathbb{C}))$. Let V_i be an irreducible representation of highest weight i . Then $i_n : V_n \rightarrow V_1^{\otimes n}$ is defined by

$$i_n(v^m) = \left[\begin{array}{c} n \\ \frac{n-m}{2} \end{array} \right]^{-1} \sum_{s, |s|=m} q^{\|s\|_-} v^{s_1} \otimes \dots \otimes v^{s_n}$$

and $\pi_n : V_1^{\otimes n} \rightarrow V_n$ is defined by

$$\pi_n(v^{s_1} \otimes v^{s_2} \otimes \dots \otimes v^{s_n}) = q^{-\|s\|_+} v^{|s|}$$

where $s = (s_1, s_2, \dots, s_n)$, $s_i = \pm 1$, $|s| = \sum s_i$ and $\|s\|_+ = \sum_{i < j} \{s_i > s_j\}$, $\|s\|_- = \sum_{i > j} \{s_i > s_j\}$ and $\{a > b\} = 1$ if $a > b$, and 0 otherwise.

Then the composition $i_n \circ \pi_n$ is called a *Jones-Wenzl projector*, denoted by p_n . It has the following properties 1) it is an idempotent 2) $p_n e_i = 0 = e_i p_n$ where e_i is a U-turn from the i -th to the $i + 1$ -th string as in the following figures.

$$n \text{---} \boxed{} \text{---} n \text{---} \boxed{} \text{---} n = n \text{---} \boxed{} \text{---} n \quad , \quad n \text{---} \boxed{} \text{---} \begin{matrix} k \\ n - k - 2 \end{matrix} = 0$$

We can generalize the second property as follows: if we attach a web with a cut path with less weight, then it is zero. Then we can axiomatize these two properties to define generalized Jones-Wenzl projectors for any simple Lie algebra. Kuperberg [Kup96] proved that Jones-Wenzl projectors exist for simple Lie algebra of rank 2 and he called *clasps* (sometimes they are called *magic weaving elements* or *boxes*). Here, we will call them clasps. For $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$, it is known that we can inductively expand it as in equation 2.1. For advanced calculations, the single clasp expansion in equation 2.2 is very useful and has been used in [Kho97] for some beautiful results. By symmetry, there are four different positions for the single clasp expansion depending on where the clasp of weight $n - 1$ is located. For equation 2.2, the clasp is located at the southwest corner, which will be considered the standard expansion, otherwise, we will state the location of the clasp.

$$\begin{array}{c} n \\ | \\ \boxed{} \\ | \\ n \end{array} = \sum_{i=1}^n a_i \begin{array}{c} i \quad 1 \\ | \quad | \\ | \quad | \\ | \quad | \\ | \quad | \\ \boxed{} \\ | \\ n - 1 \end{array} \quad (2.2)$$

Proposition 2.1 *The coefficients in equation 2.2 are*

$$a_i = \frac{[n + 1 - i]}{[n]}.$$

Proof: By attaching a U turn at consecutive strings to the top, we have the following $n - 1$ equations.

$$a_{n-1} - [2]a_n = 0.$$

For $i = 1, 2, \dots, n-2$,

$$a_i - [2]a_{i+1} + a_{i+2} = 0.$$

One can see that these equations are independent. By attaching the clasp of weight n to the bottom of every web in equation 2.2, we get $a_1 = 1$ by the properties of a clasp. This process is called a *normalization*. Then we check the answer in the proposition satisfies these equations. Since these webs in equation 2.2 form a basis, these coefficients are unique. \square

2.2.2 Applications of Single Clasp Expansions for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$

We can easily prove the following propositions using the single clasp expansion of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$. Let $a + b = c + d$ and $b = \min\{a, b, c, d\}$.

$$\begin{array}{c} a \quad b \\ | \quad | \\ \text{---} \\ | \quad | \\ c \quad d \end{array} = \sum_{k=0}^b a_k \begin{array}{c} a \quad b \\ | \quad | \\ \text{---} \quad \text{---} \\ | \quad | \quad | \quad | \\ c \quad d \end{array} \quad (2.3)$$

Proposition 2.2 *The coefficients in equation 2.3 are*

$$a_k = \frac{[c]![b]![a+b-k]}{[c-k]![b-k]![k]![a+b]}.$$

Proof: We induct on $a + b$. If $a + b = 1$, it is clear. Without loss of generality, we assume that $a \geq b$. Denote the diagram corresponding to the coefficient a_k in the right hand side of equation 2.3 by $D(k)$. By applying a single clasp expansion for

the clasp of weight $a + b$, then a single clasp expansion of the clasp located at the northeast corner, we get

$$\begin{array}{c} a \\ | \\ \text{---} \\ | \\ c \end{array} \begin{array}{c} b \\ | \\ \text{---} \\ | \\ d \end{array} = \begin{array}{c} a \quad b \\ | \quad | \\ \text{---} \quad \text{---} \\ | \quad | \\ c \quad d-1 \\ | \quad | \\ \text{---} \quad \text{---} \\ | \quad | \\ c \quad d \end{array} - \frac{[a][c]}{[a+b][a+b-1]} \begin{array}{c} a \quad b \\ | \quad | \\ \text{---} \quad \text{---} \\ | \quad | \\ c-1 \quad d-1 \\ | \quad | \\ \text{---} \quad \text{---} \\ | \quad | \\ c \quad d \end{array}$$

By induction, the right side equals

$$\begin{aligned}
 & \sum_{k=0}^{b-1} \frac{[c]![b-1]![a+b-1-k]!}{[c-k]![b-1-k]![k]![a+b-1]!} D(k) \\
 & + \frac{[a][c]}{[a+b][a+b-1]} \sum_{k=0}^{b-1} \frac{[c-1]![b-1]![a+b-2-k]!}{[c-1-k]![b-1-k]![k]![a+b-2]!} D(k+1) \\
 & = D(0) + \sum_{k=1}^{b-1} \left(\frac{[c]![b-1]![a+b-1-k]!}{[c-k]![b-1-k]![k]![a+b-1]!} \right. \\
 & + \frac{[a][c]}{[a+b][a+b-1]} \frac{[c-1]![b-1]![a+b-2-(k-1)]!}{[c-1-(k-1)]![b-1-(k-1)]![k-1]![a+b-2]!} \Big) D(k) \\
 & + \frac{[a][c]}{[a+b][a+b-1]} \frac{[c-1]![b-1]![a+b-2-(b-1)]!}{[c-1-(b-1)]![b-1-(b-1)]![b-1]![a+b-2]!} D(b) \\
 & = D(0) + \sum_{k=1}^{b-1} \frac{[c]![b]![a+b-k]!}{[c-k]![b-k]![k]![a+b]!} \left(\frac{[b-k][a+b] + [k][a]}{[b][a+b-k]} \right) D(k) \\
 & + \frac{[c]![b]![a]!}{[c-b]![0]![b]![a+b]!} D(b) = \sum_{k=0}^b \frac{[c]![b-1]![a+b-k]!}{[c-k]![b-k]![k]![a+b]!} D(k).
 \end{aligned}$$

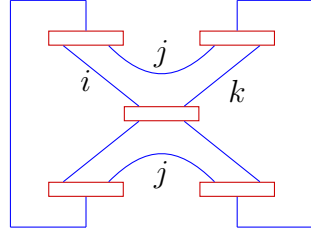
We use a well-known identity for quantum integers,

$$[m+r][n+r] = [m][n] + [m+n+r][r]$$

in the 6-th line of the above equation with $n = -a, m = -k$ and $r = a + b$. \square

Next we look at the trihedron coefficient (or $3j$ symbol) [Lic72] [MV94] [Tur94].

Proposition 2.3 *The trihedron coefficient is*



$$= (-1)^{i+j+k} \frac{[i+j+k+1]![i]![j]![k]!}{[i+j]![j+k]![i+k]!}.$$

Proof: The idea of the proof is identical to the previous proposition. We induct on $i+j+k$. If $i+j+k = 1$, it is just a circle, so its value is

$$-[2] = (-1)^1 \frac{[2]![1]![0]![0]!}{[1]![1]![0]!}.$$

We apply a single clasp expansion for the clasp of weight $i+k$, then another single clasp expansion, for which the clasp is located at the northeast corner. By induction, we have

$$\begin{aligned} &= -\frac{[j+k+1]}{[j+k]} (-1)^{i+j+k-1} \frac{[i+j+k]![i]![j]![k-1]!}{[i+j]![i+k-1]![j+k-1]!} \\ &+ \frac{[i][i]}{[i+k][i+k-1]} (-1)^{i+j+k-1} \frac{[i+j+k]![i-1]![j+1]![k-1]!}{[i+j]![i+k-2]![j+k]!} \\ &= (-1)^{i+j+k} \frac{[i+j+k+1]![i]![j]![k]!}{[i+j]![i+k]![j+k]!} \left(\frac{[i+k][j+k]}{[i+j+k+1][k]} - \frac{[i][j+1]}{[i+j+k+1][k]} \right) \\ &= (-1)^{i+j+k} \frac{[i+j+k+1]![i]![j]![k]!}{[i+j]![i+k]![j+k]!} \end{aligned}$$

□

2.3 Single Clasp Expansion for $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$

A complete set of relations for skein theory of $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$ as given in equations 2.4 2.5 2.6 was found in [Kup96]. There is a relation for every *elliptic face*, a face with less than 6 edges. We call the relation 2.6 a *rectangular relation* and the first(second) shape

in the right side of the equality is called a *horizontal*(*vertical*, respectively) *splitting*. For several reasons, such as positivity and integrality, we use $-[2]$ in relation 2.5 but one can use a quantum integer $[2]$ and get an independent result. If one uses $[2]$, one can rewrite all results in here by multiplying each trivalent vertex by the complex number i .

$$\begin{array}{c} \circlearrowright \end{array} = [3] \quad (2.4)$$

$$\begin{array}{c} \leftarrow \circlearrowleft \rightarrow \end{array} = -[2] \begin{array}{c} \leftarrow \end{array} \quad (2.5)$$

$$\begin{array}{c} \nearrow \quad \rightarrow \quad \nwarrow \\ \leftarrow \quad \square \quad \rightarrow \\ \swarrow \quad \leftarrow \quad \searrow \end{array} = \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} + \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} + \begin{array}{c} \curvearrowleft \\ \curvearrowleft \end{array} \quad (2.6)$$

A clasp for $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$ can be defined axiomatically: 1) it is an idempotent and 2) if we attach a U turn or a Y , it becomes zero. An explicit definition of clasps for $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$ can be found in [Kup96].

First we look at a single clasp expansion of the clasp of weight $(a, 0)$ where the weight (a, b) stands for $a\lambda_1 + b\lambda_2$ and λ_i is a fundamental dominant weight of $\mathfrak{sl}(3, \mathbb{C})$. Each directed edge represents V_{λ_i} , the fundamental representation of the highest weight λ_i . We might use the notation $+$, $-$ for $V_{\lambda_1}, V_{\lambda_2}$ but it should be clear.

We recall the usual partial ordering of the weight lattice of lattice of $\mathfrak{sl}(3, \mathbb{C})$ as

$$\begin{aligned} a\lambda_1 + b\lambda_2 &\succ (a+1)\lambda_1 + (b-2)\lambda_2 \\ a\lambda_1 + b\lambda_2 &\succ (a-2)\lambda_1 + (b+1)\lambda_2. \end{aligned}$$

A *cut path* is a path which is transverse to strings of web, and the weight of a cut pass is (a, b) if it passes a strings decorated by V_{λ_1} and b strings decorated by V_{λ_1} .

2.3.1 Single Clasp Expansions of a Clasp of Weight $(n, 0)$ and $(0, n)$

A basis for the single clasp expansion is given in equation 2.7. If we attach a Y on the top of webs in the equation 2.7, there is at least one elliptic face on which we can apply our relations. This process gives us exactly the same equations we got for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$. Thus, we can easily establish proposition 2.4. Moreover, this single clasp expansion holds for any $\mathcal{U}_q(\mathfrak{sl}(n, \mathbb{C}))$ where $n \geq 4$ because $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$ is naturally embedded in $\mathcal{U}_q(\mathfrak{sl}(n, \mathbb{C}))$. Later we will mention the importance of this fact. As same as for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$, there are four different positions for the single clasp expansions for $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$ so we use the same convention we used for single clasp expansion for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$.

$$\text{Clasp}(n, 0) = \sum_{i=1}^n a_i \text{Clasp}(i, 1) \quad (2.7)$$

Proposition 2.4 *The coefficients in equation 2.7 are*

$$a_i = \frac{[n+1-i]}{[n]}.$$

Also, we can easily find a single clasp expansion of the clasp of weight $(0, b)$ by reversing arrows.

2.3.2 Single Clasp Expansions of a Non-segregated Clasp of Weight (a, b)

The most interesting case is a single clasp expansion of the clasp of weight (a, b) . First of all, we find the dimension of $\text{Inv}(V_{\lambda_1}^{\otimes a+1} \otimes V_{\lambda_2}^{\otimes b} \otimes V_{(b-1)\lambda_1+a\lambda_2})$ in lemma 2.5.

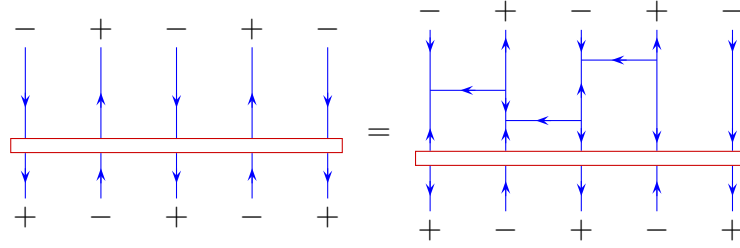
Lemma 2.5

$$\dim(\text{Inv}(V_{\lambda_1}^{\otimes a+1} \otimes V_{\lambda_2}^{\otimes b} \otimes V_{(b-1)\lambda_1+a\lambda_2})) = \begin{cases} a & \text{if } b = 0 \\ (a+1)b & \text{if } b > 0 \end{cases}$$

Proof: For $b = 0$, the result follows from proposition 2.4. We induct on b to show the multiplicity of $V_{a\lambda_1+(b-1)\lambda_2}$, $V_{(a+1)\lambda_1+b\lambda_2}$ and $V_{(a-1)\lambda_1+(b+1)\lambda_2}$ in the decomposition of $V_{1,0}^{\otimes a+1} \otimes V_{0,1}^{\otimes b}$ into irreducible representations are $(a+1)b$, 1 and a respectively. By a simple application of Schur's Lemma, we find that the dimension of $\text{Inv}(V_{\lambda_1}^{\otimes a+1} \otimes V_{\lambda_2}^{\otimes b} \otimes V_{(b-1)\lambda_1+a\lambda_2})$ is equal to the multiplicity of $V_{a\lambda_1+(b-1)\lambda_2}$ in the decomposition of $V_{\lambda_1}^{\otimes a+1} \otimes V_{\lambda_2}^{\otimes b}$ into irreducible representations. \square

Lemma 2.5 work for any A_n where $n \geq 2$ by replacing λ_2 by λ_n . Moreover the single clasp expansion of the clasp of weight $a\lambda_1 + b\lambda_n$ as in equation 2.7 is also true for any $\mathcal{U}_q(\mathfrak{sl}(n, \mathbb{C}))$ where $n \geq 3$.

We need a set of basis webs with nice rectangular order, but we can not find one in the general case. Even if one find a basis, they have many hexagonal faces which make it very difficult to get numerical relations. So we start from an alternative, *non-segregated* clasp. A non-segregated clasp is obtained from the segregated clasp by attaching a sequence of H 's until we get the desired shape of edge orientations. Fortunately, there is a canonical way to find by putting H from the leftmost string of weight λ_2 or $-$ until it reach to the desired position. In the following lemma, we will show the non-segregated clasp is well defined. The following figure is an example of a non-segregated clasp of weight $(2, 3)$ and how to obtain it from a segregated clasp of weight $(2, 3)$.



Lemma 2.6 *Non-segregated clasps are well-defined.*

Proof: Let α be a sequence of H 's which induce the same non-segregated clasp. We find the first string from the leftmost of sign $-$ which does move to right. If there is no such a string, then the clasp is canonical. Otherwise, there exist two consecutive H 's which can be removed by the horizontal splitting because it had to reach the desired position. We induct on the length of the sequence and it completes the proof.

□

We also find that non-segregated clasps satisfy two properties of segregated clasps.

Lemma 2.7 1) *Two consecutive non-segregated clasps is equal to a non-segregated clasp.*

2) *If we attach a web to a non-segregated clasp and if it has a cut path of which weight is less than the weight of the clasp, then it is zero.*

Proof: Since there is two consecutive non-segregated clasps, we see that the bottom end of the upper non-segregated clasp and the top end of the lower non-segregated clasp are the same non-segregated. Since non-segregated clasp does not depend on the choice of order of attaching H 's, we fix the canonical one for the both side then we can see that one is the other's inverse, the inverse is just the horizontal reflection. Therefore, all the H 's in the middle cancel out and standard clasps are idempotents.

The second part is obvious because adding H 's does not change the weight of the minimal path so we can change to a segregated clasp by adding more H 's. Then the clasp becomes zero.

□

The following equation 2.8 is a single clasp expansion of a non-segregated clasp of weight (a, b) . Let us denote the web corresponding to the coefficient $a_{i,j}$ by $D_{i,j}$. These webs form a basis because there are no applicable relations.

$$\begin{array}{c} b \quad a+1 \\ \vdots \quad \vdots \\ \text{---} \\ \vdots \quad \vdots \end{array} = \sum_{i=1}^b \sum_{j=0}^a a_{i,j} \begin{array}{c} 1 \quad i-1 \quad j \quad 1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ \text{---} \\ \vdots \quad \vdots \quad \vdots \quad \vdots \end{array} \quad (2.8)$$

Kuperberg [Kup96] showed that for a fixed boundary, interior can be filled by a cut out from the hexagonal tiling of the plane with the given boundary. For our cases, there are two possible fillings but we use the maximal cut out of the hexagonal tiling. We draw examples of the case $i = 6, j = 5$ and the first one in equation 2.9 is not maximal cut out and the second one is the maximal cut out which fits to the left rectangle and the last one is the maximal cut out which fits to the right rectangle as the number indicates in equation 2.8.

$$\begin{array}{c} \text{---} \\ \vdots \quad \vdots \end{array}, \begin{array}{c} \text{---} \\ \vdots \quad \vdots \end{array}, \begin{array}{c} \text{---} \\ \vdots \quad \vdots \end{array} \quad (2.9)$$

(1) (2)

Theorem 2.8 *The coefficients in equation 2.8 are*

$$a_{i,j} = \frac{[b-i+1]}{[b]} \frac{[b+j+1]}{[a+b+1]}.$$

Proof: As usual, we attach a Y or a U turn to find one exceptional and three types of equations as follow.

$$[3]a_{1,0} - [2]a_{1,1} - [2]a_{2,0} + a_{2,1} = 0.$$

Type I : For $j = 0, 1, \dots, a$,

$$a_{b-1,j} - [2]a_{b,j} = 0.$$

Type II : For $i = 1, 2, \dots, b-2$ and $j = 0, 1, \dots, a$.

$$a_{i,j} - [2]a_{i+1,j} + a_{i+2,j} = 0.$$

Type III : For $i = 1, 2, \dots, b$ and $j = 0, 1, \dots, a-2$.

$$a_{i,j} - [2]a_{i,j+1} + a_{i,j+2} = 0.$$

We establish the following lemma 2.9 first.

Lemma 2.9 *Let $a_{1,0} = x$, then the coefficients in the equation 2.8 is*

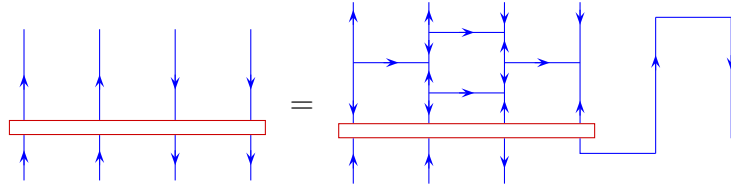
$$a_{i,j} = \frac{[b-i+1][b+j+1]}{[b][b+1]}x.$$

Proof: First we can see that the right side of equation 2.8 is a basis for a single clasp expansion because its number of the webs is equal to the dimension as in lemma 2.5 and none of these webs has any faces. Second we find that these equations has at least $(a+1)b$ independent equations. Then we plug in these coefficients to equations to check that they are the right coefficients. \square

Usually we normalize one basis web in the expansion to get a known value. But we can not normalize for this expansion yet because it is not a segregated clasp. Thus we use lemma 2.12 to find that the coefficient of $a_{1,a}$ is 1. Then, we get $x = \frac{[b+1]}{[a+b+1]}$ and it completes the proof of the theorem. \square

2.3.3 Double Clasp Expansion of a Segregated Clasp of Weight (a, b)

Now we study a single and a double clasps expansion of a segregated clasp of weight (a, b) . We will start with an example, a single clasp expansion of a clasp of weight $(2, 2)$. To apply theorem 2.8, we add some H's to change the segregated clasp to a non-segregated clasp.



Then we expand the clasp. In the following equation 2.10, we will omit the direction of edges unless there is an ambiguity.

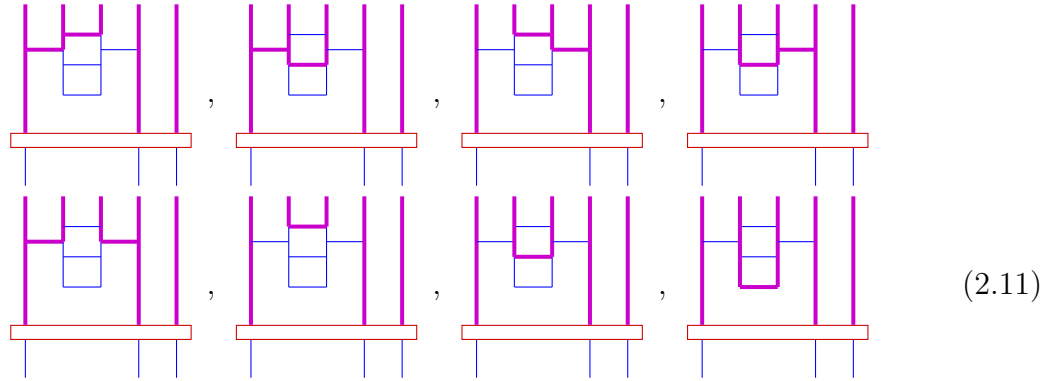
$$\begin{aligned}
 & \frac{[3]}{[5]} \text{ (web 1)} + \frac{[4]}{[5]} \text{ (web 2)} + \frac{[5]}{[5]} \text{ (web 3)} \\
 & + \frac{[1][2]}{[3][5]} \text{ (web 4)} + \frac{[1][2]}{[4][5]} \text{ (web 5)} + \frac{[1][2]}{[5][5]} \text{ (web 6)} \quad (2.10)
 \end{aligned}$$

These webs can be expanded using relations. For example, the first one can be expanded by equation 2.6 as

$$\text{ (web 1)} = \text{ (web 1a)} - [2] \text{ (web 1b)}$$

For some small cases, we can expand this way but it will be difficult to manage all possible expansions. An other way to look at this expansion is to use paths : since

there are five points on the top and three points right above the clasp and these three points have to be connected to points on the top (otherwise, we have a cut path with weight less than $(2, 1)$ which makes the web zero), we have two Y 's or one U turn. We first find all possible disjoint, monotone(except at Y 's) paths connecting these points. For the web on above example, there are eight possibilities as follows.



If we examine them to determine whether it will appeared in the actual expansion, the first two in the second row appear but the rest of them do not. One can see that U turn can appear only once at the very top.

$$\begin{aligned}
& \left[\text{Diagram with 5 vertical lines and a red clasp} \right] = \frac{[3]}{[5]} \left[-[2] \left[\text{Diagram 1} \right] - [2] \left[\text{Diagram 2} \right] \right] \\
& + \frac{[4]}{[5]} \left[-[2] \left[\text{Diagram 3} \right] + \left[\text{Diagram 4} \right] + \left[\text{Diagram 5} \right] \right] \\
& + \frac{[5]}{[5]} \left[\left[\text{Diagram 6} \right] + \left[\text{Diagram 7} \right] \right] \\
& + \frac{[1][2]}{[3][5]} \left[\left[\text{Diagram 8} \right] + \left[\text{Diagram 9} \right] + \left[\text{Diagram 10} \right] \right] \\
& + \frac{[1][2]}{[4][5]} \left[\left[\text{Diagram 11} \right] + \left[\text{Diagram 12} \right] + \left[\text{Diagram 13} \right] \right] \\
& + \frac{[1][2]}{[5][5]} \left[\left[\text{Diagram 14} \right] + \left[\text{Diagram 15} \right] \right]
\end{aligned} \tag{2.12}$$

Thus, we get a single clasp expansion of a clasp of weight $(2, 2)$ as follows.

$$\begin{aligned}
 & \text{Diagram} = \text{Diagram} + \frac{[1]}{[2]} \text{Diagram} - \frac{[1]}{[2][5]} \text{Diagram} \\
 & - \frac{[2]}{[5]} \text{Diagram} - \frac{[1]}{[5]} \text{Diagram} - \frac{[1]}{[5]} \text{Diagram} \quad (2.13)
 \end{aligned}$$

By attaching $(2, 1)$ clasps on the left top of every web in the right side of equation 2.13, we get the following double clasps expansion of the clasp of weight $(2, 2)$.

$$\text{Diagram} = \text{Diagram} + \frac{[1]}{[2]} \text{Diagram} - \frac{[1]}{[5]} \text{Diagram} \quad (2.14)$$

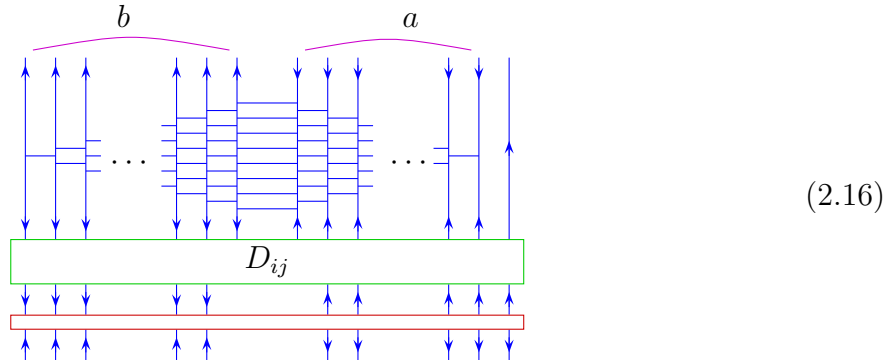
Unfortunately, there is no particular order we can put for these basis webs for single clasp expansions. But for the double clasp expansion, we can generalize the equation 2.14 as follow. In equation 2.15, the green box between two clasps is the unique cut out from the hexagonal tiling with the given boundary as we have seen in Figure 2.9. For equation 2.15 we assume that $a \geq b \geq 1$.

$$\begin{aligned}
 & \text{Diagram} = \text{Diagram} + \alpha \text{Diagram} + \beta \text{Diagram} \quad (2.15)
 \end{aligned}$$

Theorem 2.10 *The coefficients in equation 2.15 are $\alpha = \frac{[b-1]}{[b]}$, $\beta = -\frac{[a]}{[b][a+b+1]}$. We assume that $[0] = 0$ for α .*

Proof: It follows from lemma 2.11 and lemma 2.12 that $\alpha = a_{2,a}$ and $\beta = a_{2,a-1} - [2]a_{1,a-1} + a_{1,a}$. \square

To prove two key lemmas, we generalize the idea of paths in the first example. First, we attach H 's as in figure 2.16 to all basis webs in equation 2.8 each of basis web is denoted by D_{ij} . After attaching H 's as in figure 2.16, the resulting web is denoted by \tilde{D}_{ij} .

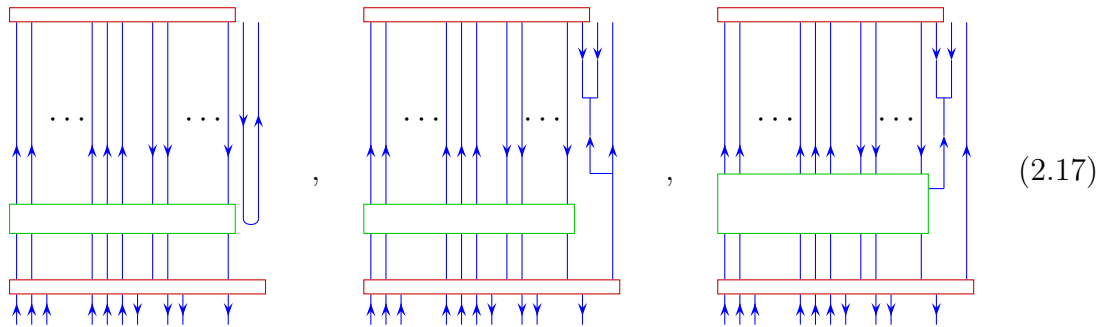


As we have seen in the example, $\tilde{D}_{i,j}$ is not a basis web because it contains some elliptic faces. If we decompose each $\tilde{D}_{i,j}$ into a linear combination of some webs which have no elliptic faces, then the union of all these resulting webs actually forms a basis. Let us prove that these webs actually form a basis which will be denoted by $D'_{i',j'}$. As vector spaces, this change, adding H 's, induces an isomorphism between two web spaces. Its matrix representation with respect to these web bases $\{D_{i,j}\}$ and $\{D'_{i',j'}\}$ is an $(a+1)b \times (a+1)b$ matrix whose entries are 0, 1 or $-[2]$. In general, we will not be able to write this matrix because there are many nonzero entries in every columns and rows. But we know that the determinant of this matrix is $\pm[2]^{ab}$ because each one H contributes $\pm[2]$ depending on the directions.

Since $\tilde{D}_{i,j}$ is not a basis web, to find a single clasp expansion, we might have to use relations to find its linear expansion into a new web basis $D'_{i',j'}$. In general this might not be done. If we just limit ourself to a double clasp expansion, We could use the

idea of paths as we demonstrated in the example. Let us formally define it, a *stem* of a web. Geometrically it is transversal to cut paths. From $\tilde{D}_{i,j}$, we see that there are $a+b+1$ points on top but only $a+b-1$ lines right above the clasp. Because of one of properties of the clasp of weight $(a, b-1)$: if we have a cut path of weight which is less than $(a, b-1)$, then the web becomes zero, we must have $a+b-1$ vertical lines which connect top $a+b+1$ nodes to the clasp of weight $(a, b-1)$ for non-vanishing webs after applying relations. It is clear that these connecting lines should be mutually disjoint, otherwise, we will have a cut path with weight less than $(a, b-1)$. A *stem* of a web is a disjoint union of lines as we described. Unfortunately some of stems do not arise all cases because it may not be obtained by removing elliptic faces. If a stem appears, we call it an *admissible stem*. For single clasp expansion, finding all these stems will be more difficult than an expansion by relations but for the double clasp expansion of segregated clasps, there are only few possible admissible stems whose coefficient is nonzero.

Lemma 2.11 *After attaching a clasp of weight $(a, b-1)$ to top of webs $\tilde{D}_{i,j}$ from equation 2.16, the only 3 non-vanishing shapes are those in Figure 2.17.*



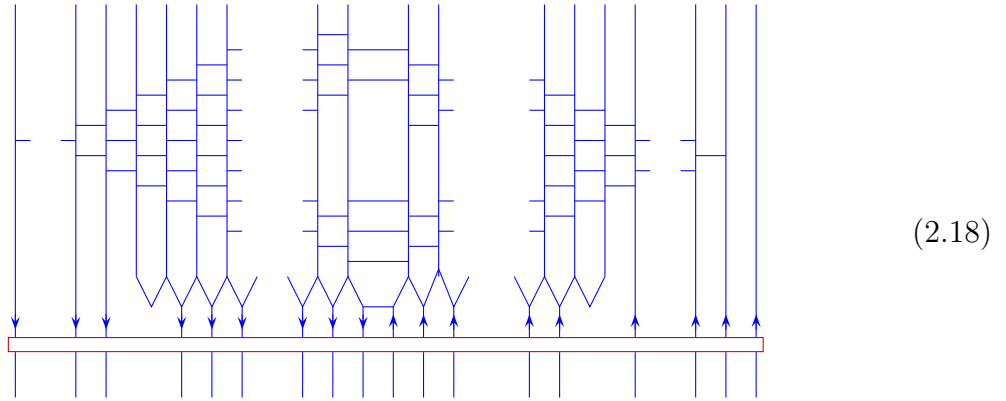
Proof: From $\tilde{D}_{i,j}$ we see that there are $a+b+1$ lines on top and $a+b-1$ lines right above the clasp. If we repeatedly use the rectangular relation as in equation 2.6, we can push up the Y 's so that there are either two Y 's or one U shape at the top. It

is possible to have two adjacent Y 's which appear in the second and third figures in Figure 2.17 but a U turn can appear in only two places because of the orientation of edges. If we attach the $(a, b - 1)$ clasp to the top of the resulting web from the left and U or Y shape appear just below it, the web becomes zero. Therefore only these three webs do not vanish. \square

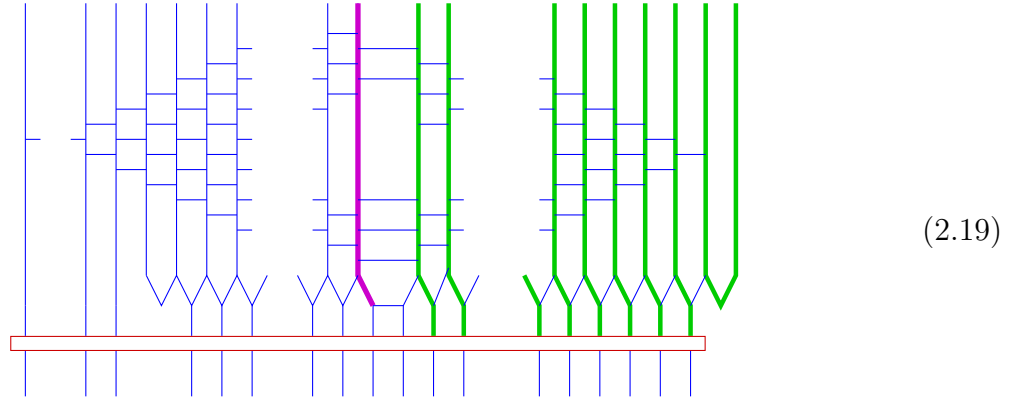
For the next lemma, we will find all $\tilde{D}_{i,j}$'s which can be transformed to each of the figures in Figure 2.17.

Lemma 2.12 *Only $\tilde{D}_{1,a}(\tilde{D}_{2,a})$ can be transformed to the first(second, respectively) shape in Figure 2.17. Only the three webs, $\tilde{D}_{1,a-1}$, $\tilde{D}_{1,a}$ and $\tilde{D}_{2,a-1}$ can be transformed to the last shape. Moreover, all of these transformations use only rectangular relations as in equation 2.6 except for the transformation from $\tilde{D}_{1,a-1}$ to the third figure uses one loop relation in equation 2.5.*

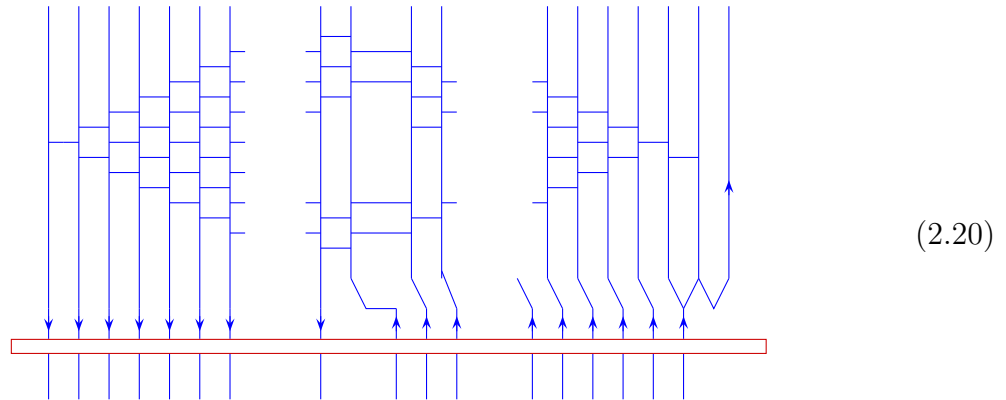
Proof:



For the first shape in figure 2.17, it is easy to see that we need to look at $\tilde{D}_{i,a}$, for $i = 1, 2, \dots, b$, otherwise the last two strings can not be changed to the shape with a U turn as shown in figure 2.18.

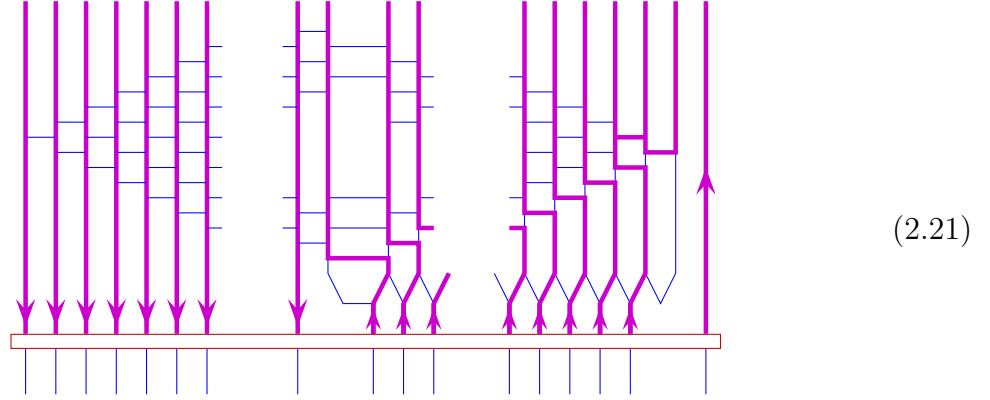


Now we look at the $D_{i,a}$ where $i > 1$ as in figure 2.19. Since we picked where the U turn appears already, one can find a candidate for a stem as thick and shaded (green in color) line from the right hand side but we can not finish because the purple string can not be join to the bottom clasp without being zero(it will force to have a generator caps off). So only nonzero admissible stems should be obtained from $\tilde{D}_{1,a}$. We split the rectangle(only one in the middle) vertically(horizontal splitting vanishes immediately) and it creates another rectangle at right top side of previous place. We have to split vertically except in the last step, for this rectangle, as in the figure 2.20, both splits do not vanish. The vertical split gives us the first shape figure 2.17 and the horizontal split gives the third shape in figure 2.17.

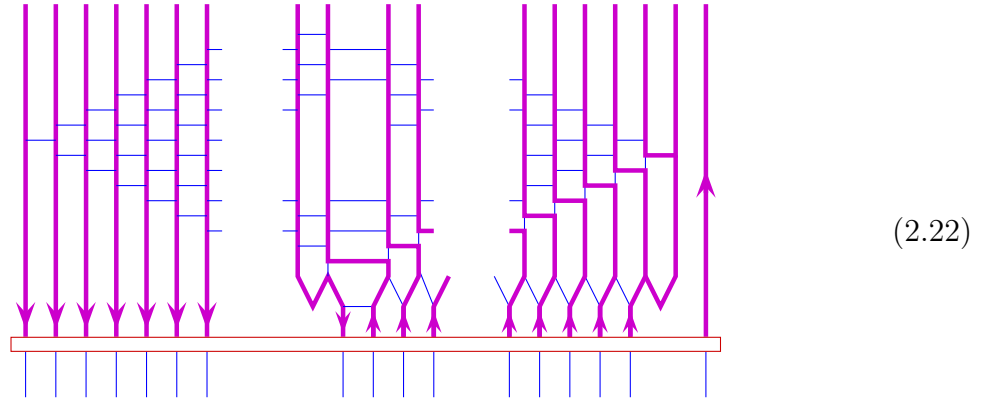


A similar argument works for the second one in figure 2.17. The third figure

in 2.17 is a little subtle. First one can see that none of the $\tilde{D}_{i,j}$ work if either $i > 2$ or $j < a - 1$. Thus, we only need to check $\tilde{D}_{1,a-1}$, $\tilde{D}_{1,a}$, $\tilde{D}_{2,a-1}$ and $\tilde{D}_{2,a}$ but we already know about $\tilde{D}_{1,a}$, $\tilde{D}_{2,a}$. The following figure 2.21 shows the nonzero admissible stem for $\tilde{D}_{1,a-1}$. As usual, we draw a stem as a union of thick and purple lines.



The following figure 2.22 shows the nonzero admissible stem for $\tilde{D}_{2,a-1}$.



Note that the last figure has one loop which contributes $-[2]$. This completes the proof of lemma. \square

Corollary 2.13

$$\begin{aligned}
& \begin{array}{c} a \\ \uparrow \\ \text{---} \\ \uparrow \\ a \end{array} \bigcirc 1 = \frac{[a+3]}{[a+1]} \begin{array}{c} a \\ \uparrow \\ \text{---} \\ \uparrow \\ a \end{array} \\
& \begin{array}{cc} a & b \\ \uparrow & \downarrow \\ \text{---} & \\ \uparrow & \downarrow \\ a & b \end{array} \bigcirc 1 = \frac{[b+2][a+b+3]}{[b+1][a+b+2]} \begin{array}{cc} a & b \\ \uparrow & \downarrow \\ \text{---} & \\ \uparrow & \downarrow \\ a & b \end{array} \\
& \begin{array}{c} a \\ \uparrow \\ \bigcirc \\ \downarrow \end{array} \text{---} = \frac{[a+2][a+1]}{[2]} \\
& \begin{array}{cc} \bigcirc & \bigcirc \\ a & b \end{array} \text{---} = \frac{[a+1][b+1][a+b+2]}{[2]} \quad (2.23)
\end{aligned}$$

Proof: After using a double clasps expansion one can get the first two equalities with a simple calculation. The next two follow from the previous two by induction.

□

We will apply theorem 2.8 to derive the coefficients in equations 2.24 and 2.26. The expansion in the proposition 2.24 is known [Kup96], which is only previously known expansion formula for a segregated clasp of weight (a, b) and it was used to find quantum $su(3)$ invariants in [OY]. Our proof using single clasp expansion will be used for trihedron coefficients.

$$\begin{array}{c} a \quad b \\ \downarrow \quad \downarrow \\ \boxed{} \\ \uparrow \quad \uparrow \\ a \quad b \end{array} = \sum_{k=0}^{\text{Min}(a,b)} a_k \begin{array}{c} a \quad b \\ \downarrow \quad \downarrow \\ \boxed{} \quad \boxed{} \\ \uparrow \quad \uparrow \\ a \quad b \end{array} \quad (2.24)$$

Proposition 2.14 *The coefficients in equation 2.24 is*

$$a_k = (-1)^k \frac{[a]![b]![a+b-k+1]!}{[a-k]![b-k]![k]![a+b+1]!}.$$

Proof: Let me denote that a basis web in the right side of equation 2.24 by $D(k)$ which corresponding to the coefficient a_k . We induct on $a+b$. It is clear for $a=0$ or $b=0$. If $a \neq 0 \neq b$ then we use a segregated single clasp expansion of weight (a, b) in the middle. Even if we do not use entire single clasp expansion of segregated clasp, once we attach $(a, 0), (0, b)$ clasps on the top, there is only two surviving web which are one with one U turn. One of resulting webs has some H 's as in Figure 2.25 but if we push them down to $(a, b-1)$ clasp, it becomes a non-segregated clasp.

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} - \frac{[a]}{[a+b+1]} \text{Diagram 3} \\
 & = \text{Diagram 4} - \frac{[a][a+1]}{[a+b+1][a+b]} \text{Diagram 5} \quad (2.25)
 \end{aligned}$$

We can find the coefficient using the same argument using stems and it is

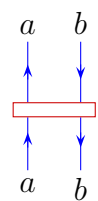
$$-[2]a_{1,a-b} + a_{a,a-b+1} + \sum_{i=2}^b (a_{i,a-b+i-2} - [2]a_{i,a-b+i-1} + a_{i,a-b+i}) = -\frac{[a]}{[a+b+1]}$$

because $a_{i,a-b+i-2} - [2]a_{i,a-b+i-1} + a_{i,a-b+i} = 0$ for all $i = 2, 3, \dots, b$. Then we attach some H 's to make the middle clasp as a non-segregated clasp of weight $(a, b-1)$. By using a non-segregated single clasp expansion for which clasps are located at northeast corner and by the induction hypothesis, we have

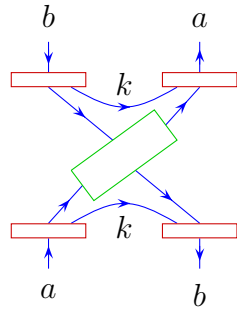
$$\begin{aligned}
&= \sum_{k=0}^{b-1} (-1)^k \frac{[a]![b-1]![a+b-k]!}{[a-k]![b-1-k]![k]![a+b]!} D(k) \\
&\quad - \frac{[a+1][a]}{[a+b+1][a+b]} \sum_{k=0}^{b-1} (-1)^k \frac{[a-1]![b-1]![a+b-1-k]!}{[a-1-k]![b-1-k]![k]![a+b-a]!} D(k+1) \\
&= 1 \cdot D(0) + \sum_{k=1}^{b-1} ((-1)^k \frac{[a]![b-1]![a+b-k]!}{[a-k]![b-1-k]![k]![a+b]!} \\
&\quad + (-1)^k \frac{[a+1]![b-1]![a+b-k]!}{[a-k]![b-1-k]![k-1]![a+b]!}) D(k) \\
&\quad - (-1)^{b-1} \frac{[a+1][a]}{[a+b+1][a+b]} \frac{[a-1]![b-1]![a]!}{[a-b]![0]![b-1]![a+b-1]!} D(b) \\
&= D(0) + \sum_{k=1}^{b-1} (-1)^k \frac{[a]![b]![a+b+1-k]!}{[a-k]![b-k]![k]![a+b+1]!} \left(\frac{[b-k][a+b+1] + [k][a+1]}{[b][a+b+1-k]} \right) D(k) \\
&\quad + (-1)^b \frac{[a]![b-1]![a+1]!}{[a-b]![0]![b-1]![a+b+1]!} D(b) \\
&= \sum_{k=0}^b (-1)^k \frac{[a]![b]![a+b+1-k]!}{[a-k]![b-k]![k]![a+b+1]!} D(k)
\end{aligned}$$

□

For equation 2.26, we assume $0 \leq a \leq b$.



$= \sum_{k=0}^b A(a, b, k)$



(2.26)

Proposition 2.15 *The coefficients $A(a, b, k)$ in equation 2.26 satisfies the following recurrence relation.*

$$A(1, 1, 1) = \frac{[2]}{[3]},$$

$$A(1, 1, 0) = 1,$$

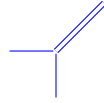
$$A(a, a, k) = \frac{[2a - k + 1]}{[2a + 1]}(A(a - 1, a, k) + A(a - 1, a, k - 1)),$$

$$A(a, a + i, k) = \frac{[2a + 1 + i - k]}{[2a + 1 + i]}A(a, a + i - 1, k).$$

Proof: Note that we assume that $A(a, b, -i) = A(a, b, a + i) = 0$ for all $i > 0$. Using a non-segregated single clasp expansion at the clasp of weight (a, b) , one standard and one with clasp in the northeast corner, we have the result with two axioms of clasps. Remark that these coefficients are not round. \square

2.4 Single Clasp Expansion for $\mathcal{U}_q(\mathfrak{sp}(4))$

It is known [Kup96] that $\mathcal{U}_q(\mathfrak{sp}(4))$ webs are generated by a single web



with the relations

$$\begin{aligned}
 \text{Circle} &= -\frac{[6][2]}{[3]} \\
 \text{Double Circle} &= \frac{[6][5]}{[3][2]} \\
 \text{Loop} &= 0 \\
 \text{Double Loop} &= -[2]^2 \\
 \text{Triangle} &= 0 \\
 \text{Crossing} - \text{Vertex} &= \text{Arc} - \text{Arc} \quad (2.27)
 \end{aligned}$$

Also it is known [Kup96] that we can define tetravalent vertex to achieve the same end as in equation 2.28. First we will find a single clasp expansion of clasps of weight $(n, 0)$ and $(0, n)$ and then use it to find coefficients of double clasps expansion of clasps of weight $(n, 0)$ and $(0, n)$. Remark that the cut weight is defined slightly different way. A cut path may cut diagonally through a tetravalent vertex, and its weight is defined as $n\lambda_1 + (k + k')\lambda_2$, where n is the number of type “1”, single strands, that it cuts, k is the number of type “2”, double strands, that it cuts, and k' is the number of tetravalent vertices that it bisects. And there is a natural partial ordering of the B_2 weight lattice given by

$$\begin{aligned}
 a\lambda_1 + b\lambda_2 &\succ (a - 2)\lambda_1 + (b + 1)\lambda_2 \\
 a\lambda_1 + b\lambda_2 &\succ (a + 2)\lambda_1 + (b - 2)\lambda_2.
 \end{aligned}$$

We will use the following shapes to find a single clasp expansion because there is an ambiguity of preferred direction in the last relation 2.27. We remark that the left side of the second equality of 2.28 is not a crossing but a vertex where four double

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \quad \diagup \\ \text{---} \\ \diagdown \quad \diagdown \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array}, \quad \begin{array}{c} \diagup \quad \diagup \\ \diagdown \quad \diagdown \end{array} = \begin{array}{c} \diagup \quad \diagup \\ \text{---} \\ \diagdown \quad \diagdown \end{array} \quad (2.28)$$
$$\begin{array}{c} n \\ | \\ \text{---} \\ | \\ n \end{array} = \sum_{i=0}^{n-1} \sum_{j=i+1}^n a_{ij} \begin{array}{c} j \quad i \\ | \quad | \\ | \quad | \\ | \quad | \\ | \quad | \\ | \quad | \\ | \quad | \\ | \quad | \\ | \quad | \\ | \quad | \\ \text{---} \\ | \\ n-1 \end{array} \quad (2.29)$$

Theorem 2.16 *The coefficients in Figure 2.29 are*

$$a_{i,j} = [2]^{i-j+1} \frac{[n+1][n-j+1][2n-2i+2]}{[n][2n+2][n-i+1]}.$$

$$\begin{aligned}
\text{Diagram 1} &= \frac{[6][2]}{[3]} \text{Diagram 2}, & \text{Diagram 3} &= -[2]^2 \text{Diagram 4} \\
\text{Diagram 5} &= -[2]^2 \text{Diagram 6} - [2][4] \text{Diagram 7} \\
\text{Diagram 8} &= [2]^2 \text{Diagram 9} + [2]^2 \text{Diagram 10}
\end{aligned} \tag{2.30}$$

Using these relations, we get the following $n - 1$ equations by adding U turns from the left to right. By capping off the generator from left to right, we have $(n - 1)^2$ equations. There are two special equations and four different shapes of equation as follows.

$$a_{n-2,n-1} + \frac{[2][6]}{[3]}a_{n-2,n} - \frac{[2][6]}{[3]}a_{n-1,n} = 0$$

$$-\frac{[2][6]}{[3]}a_{12} + \frac{[2][6]}{[3]}a_{13} + a_{23} + 1 + \frac{[2][6]}{[3]}b_2 - [2][4]b_3 = 0$$

Type I : For $i = 1, 2, \dots, n - 3$,

$$a_{i,i+1} + \frac{[2][6]}{[3]}a_{i,i+2} - [2][4]a_{i,i+3} - \frac{[2][6]}{[3]}a_{i+1,i+2} + \frac{[2][6]}{[3]}a_{i+1,i+3} + a_{i+2,i+3} = 0.$$

Type II : For $i = 0, 1, \dots, n - 2$,

$$a_{i,n-1} - [2]^2a_{i,n} = 0.$$

Type III : For $i = 0, 1, 2, \dots, n - 3$, $k = 2, 3, \dots, n - i - 1$,

$$a_{i,n-k} - [2]^2a_{i,n-k+1} + [2]^2a_{i,n-k+2} = 0.$$

Type IV : For $i = 3, 4, \dots, n$, $k = n - i + 3, n - i + 4, \dots, n$,

$$[2]^2a_{n-k,i} - [2]^2a_{n-k+1,i} + a_{n-k+2,i} = 0.$$

Then we check the answer in the proposition satisfies the equations. Since these webs in equation 2.29 form a basis, the coefficients are unique. Therefore, it completes the proof. \square

$$\begin{array}{c} n \\ | \\ \text{---} \\ | \\ n \end{array} = \begin{array}{c} n-1 \\ | \\ \text{---} \\ | \\ n-1 \end{array} + a_{12} \begin{array}{c} n-1 \\ | \\ \text{---} \\ | \\ n-2 \\ | \\ \text{---} \\ | \\ n-1 \end{array} + a_{02} \begin{array}{c} n-1 \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ n-1 \end{array} \quad (2.31)$$

Corollary 2.17 *The double clasp expansion of B_2 of type $(n, 0)$ can be obtained as in the equation 2.31 where a_{12}, a_{02} are from the Theorem 2.16.*

$$\begin{array}{c} n \\ | \\ \text{---} \\ | \\ n \end{array} = \sum_{i=0}^{n-1} \sum_{j=i+1}^n a_{ij} \begin{array}{c} j \quad i \\ | \quad | \\ \text{---} \\ | \\ n-1 \end{array} \quad (2.32)$$

Then we look for $(0, n)$ case. The main idea for $(n, 0)$ works exactly same except we replace the base as in the equation 2.32. By capping off U turns and a lower weight cap, we get the following coefficients and we can solve them successively as in Theorem 2.18. Also the equation 2.33 is useful to find the following equations.

$$\begin{array}{l}
 \begin{array}{c} \diagup \\ | \\ \text{---} \\ | \\ \diagdown \end{array} = [5] \quad , \quad \begin{array}{c} \diagup \\ | \\ \text{---} \\ | \\ \diagdown \end{array} = -[2]^2[5] \\
 \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \\ | \\ \diagdown \quad \diagup \end{array} = -[2][4] \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \\ | \\ \diagdown \quad \diagup \end{array} - [2]^2[3] \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \\ | \\ \diagdown \quad \diagup \end{array} \\
 \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \\ | \\ \diagdown \quad \diagup \end{array} = -[2][4] \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \\ | \\ \diagdown \quad \diagup \end{array} + [2]^4[3] \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \text{---} \\ | \\ \diagdown \quad \diagup \end{array} \quad (2.33)
 \end{array}$$

$$\begin{aligned}
 a_{n-2,n-1} - [5][2]^2 a_{n-2,n} + \frac{[6][5]}{[3][2]} a_{n-1,n} &= 0 \\
 -[3][2]^2 a_{n-2,n} + [5] a_{n-1,n} &= 0
 \end{aligned}$$

Type I : For $i = 0, 1, \dots, n-3$,

$$a_{i,i+1} - [5][2]^2 a_{i,i+2} + [3][2]^4 a_{i,i+3} + \frac{[6][5]}{[3][2]} a_{i+1,i+2} - [5][2]^2 a_{i+1,i+3} + a_{i+2,i+3} = 0$$

Type II : For $i = 0, 1, \dots, n-2$,

$$a_{i,n-1} - [4][2] a_{i,n} = 0$$

Type III : For $i = 0, 1, \dots, n-3$ and $j = i+1, i+2, \dots, n-2$,

$$a_{i,j} - [4][2] a_{i,j+1} + [2]^4 a_{i,j+2} = 0$$

Type IV : For $i = 0, 1, \dots, n-3$ and $j = i+3, i+4, \dots, n$,

$$[2]^4 a_{i,j} - [4][2] a_{i+1,j} + a_{i+2,j} = 0$$

Type V : For $i = 1, 2, \dots, n-2$

$$-[3][2]^2 a_{i-1,i+1} + [2]^4 a_{i-1,i+2} + [5] a_{i,i+1} - [3][2]^2 a_{i,i+2} = 0.$$

Theorem 2.18 For $n \geq 2$,

$$a_{i,j} = [2]^{2(1+i-j)} \frac{[2n+1-2i][2n-2j+2]}{[2n][2n+1]}.$$

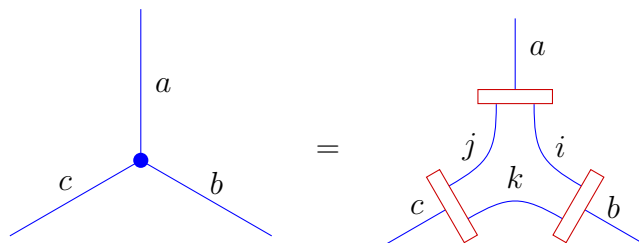
$$\text{Diagrammatic equation (2.34)} \quad (2.34)$$

Corollary 2.19 The double clasp expansion of B_2 of type $(0, n)$ can be obtained as in the equation 2.34 where a_{12}, a_{02} are from the Theorem 2.18.

Chapter 3

Trihedron Coefficients for $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$

By [Lic72] [MV94] [Tur94], we define a trivalent vertex as follow. A triple integers (a, b, c) is admissible if $a + b + c$ is even and $|a - b| \leq c \leq a + b$. This is equivalent to the following. For $\mathfrak{sl}(2, \mathbb{C})$, $\dim(\text{Inv}(V_a \otimes V_b \otimes V_c))$ is 1 if (a, b, c) is an admissible triple or 0 otherwise, where V_a is an irreducible representation of highest weight a . Given an admissible triple, we define a trivalent vertex

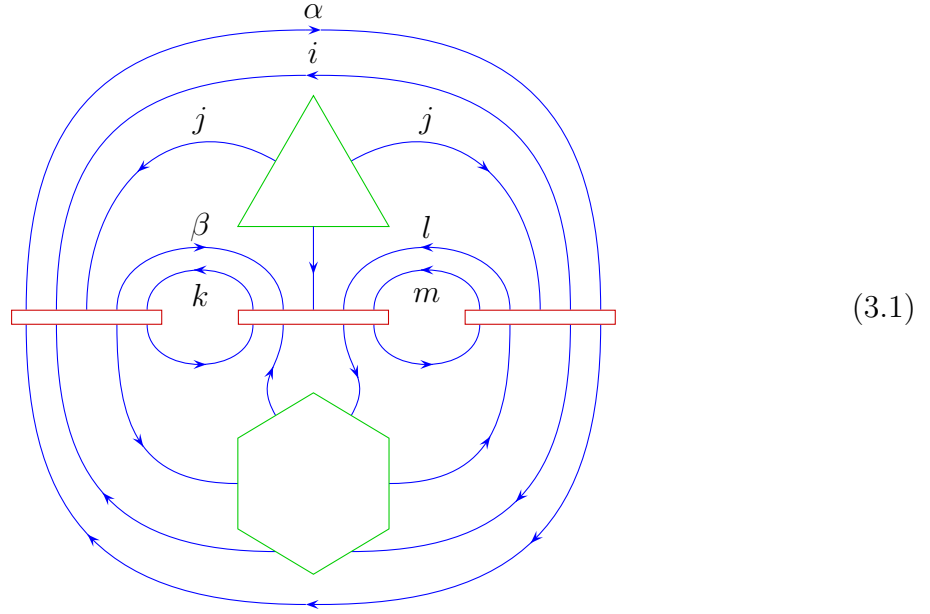


where $i = (a + b - c)/2$, $j = (a + c - b)/2$ and $k = (b + c - a)/2$. Then the trihedron coefficient for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ is known [Lic72] [MV94] [Tur94] as

$$= (-1)^{i+j+k} \frac{[i+j+k+1]![i]![j]![k]!}{[i+j]![j+k]![i+k]!}.$$

In previous chapter we found a recursive formula for generalized Jones-Wenzl projectors. So we study how we generalize trihedron coefficients to $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$. But, the definition of the trivalent vertex is a little subtle. We will prove the following statement in lemma 3.3. Let λ_1, λ_2 be the fundamental dominant weights of $\mathfrak{sl}(3, \mathbb{C})$ (mainly we will use $\mathfrak{sl}(3, \mathbb{C})$ -modules because it is known that representation theories of $\mathfrak{sl}(3, \mathbb{C})$ and $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$ are parallel: see theorem 1.1 and the representation theory of $\mathfrak{sl}(3, \mathbb{C})$ is well known in [Hum72]). Let $V_{a\lambda_1+b\lambda_2}$ be an irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$ of highest weight $a\lambda_1+b\lambda_2$. Each edge is decorated by an irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$, denoted by $V_{a_1\lambda_1+b_1\lambda_2}$, $V_{a_2\lambda_1+b_2\lambda_2}$ and $V_{a_3\lambda_1+b_3\lambda_2}$ where a_i, b_j are nonnegative integers. Let $d = \text{Min} \{a_1, a_2, a_3, b_1, b_2, b_3\}$. Then $\dim(\text{Inv}(V_{a_1\lambda_1+b_1\lambda_2} \otimes V_{a_2\lambda_1+b_2\lambda_2} \otimes V_{a_3\lambda_1+b_3\lambda_2}))$ is $d+1$ if there exist non negative integers $i, j, k, l, m, n, o, p, q$ such that $a_1 = i + p$, $a_2 = j + n$, $a_3 = k + l$, $b_1 = j + o$, $b_2 = k + m$, $b_3 = i + q$ and $0 = le^{\frac{\pi}{3}i} + me^{\frac{\pi}{6}i} + n + oe^{-\frac{\pi}{6}i} + pe^{-\frac{\pi}{3}i} - q$. Otherwise, it is zero.

If $\dim(\text{Inv}(V_{a_1\lambda_1+b_1\lambda_2} \otimes V_{a_2\lambda_1+b_2\lambda_2} \otimes V_{a_3\lambda_1+b_3\lambda_2}))$ is nonzero, we say a triple of ordered pairs $((a_1, b_1), (a_2, b_2), (a_3, b_3))$ is *admissible*. It has shown that for a fixed boundary, there are fillings which are cut outs from the the hexagonal tiling of the plane [Kup96]. A general trihedron shape is given in the following figure where $\alpha + \beta = a_1$, $i + j + k = b_1$, $k + l = a_2$, $\beta + j + m = b_2$, $i + m = a_3$ and $\alpha + j + l = b_3$. The top and the bottom part are actually the same after some modifications which we will discuss later.



So we can write trihedron coefficients as a $(d+1) \times (d+1)$ matrix. Let us denote it by $M_{\Theta}(a_1, b_1, a_2, b_2, a_3, b_3)$ or $M_{\Theta}(\lambda)$ where $\lambda = (a_1, b_1, a_2, b_2, a_3, b_3)$. Also we denotes its (i, j) entry by $\Theta_{i,j}(a_1, b_1, a_2, b_2, a_3, b_3)$ or $\Theta_{i,j}(\lambda)$. It is obvious that $M_{\Theta}(\lambda)$ is symmetric. Unfortunately the trihedron coefficient of this shape is no longer rational expression composed of monomials of quantum integers (if so, we say it to be *round*) in a simple case $((1, 1), (1, 1), (1, 1))$. So we start to look the case $a_1 = 0$. Then we have found the trihedron coefficients for the case $\alpha = \beta = 0$ and either $k = 0$ or $j = 0$ from the general shape.

Theorem 3.1 $M_{\Theta}(0, i + j, l, j + m, i + m, j + l)$ is

$$(-1)^j [j + l + 1][i + j + l + m + 2] \frac{[i + j + m + 1]![i]![j]![m]!}{[i + j]![j + m]![i + m]![2]}.$$

Theorem 3.2 $M_{\Theta}(0, i + k, k + l, m, i + m, l)$ is

$$\sum_{n=0}^{\min\{l, k, m\}} a_n \frac{[i + l + 1][i + l + m + 2]}{[i + n + 1][i + n + m + 2]} \frac{[i + k + m + 2]![k - n + 1]![i + m]![m + 1]!}{[i + k]![i + m + n]![k + m - n + 1]![2]}.$$

where

$$a_n = (-1)^n [k]^{2n} \frac{[k+l-n]![m]![k+m+l-n+1]!}{[k+l]![m-n]![k+m+l+1]}.$$

In section 2, we show all possible trihedron shapes and some properties of trihedron shapes. In section 3, we prove the main theorems.

3.1 General Shapes

Now we will look at the trihedron coefficients. The general shape is given in figure (3.1) where the weight of clasps are $a_1\lambda_1 + b_1\lambda_2$, $a_2\lambda_1 + b_2\lambda_2$ and $a_3\lambda_1 + b_3\lambda_2$ and a_i, b_j are nonnegative integers.

For $\mathfrak{sl}(2, \mathbb{C})$, $\dim(\text{Inv}(V_i \otimes V_j \otimes V_k))$ is 1 if (i, j, k) is an admissible triple or 0 otherwise, where V_i be an irreducible representation of highest weight i . So there is a unique way to fill in the triangle. But for $\mathfrak{sl}(3, \mathbb{C})$, we could have more than one ways. Thus the shape of the polygon that we are filling in might vary depending on the weight of clasps. First we will discuss how we find a general shape in figure (3.1). Instead of finding a cut out from the hexagonal tiling, we find a way to put three clasps into the hexagonal tiling. Since it bounds a polygon and we have a set of restrictions how clasps can be bent, we can change this problem to an elementary geometry problem. Since all clasps are segregated, until we reverse the direction of arrows, all possible interior angles are 60, 180 or 300. Since an Y makes the web zero, we can exclude 60. We would not count 180 because it can be seen as a subdivision. Thus it has to be 300 if we do not change the direction of arrows. When we change the direction of arrows, there are also three possible interior angle either 0, 120, 240 or 360, let us denote this angle by α_i . Since U turn makes the web zero, we can also exclude 0. Let us denote the angle between clasps by β_i . If there is a cut out bounded by three given clasps of which all a_i, b_j are nonzero, we can see the following equality

from the sum of interior angles.

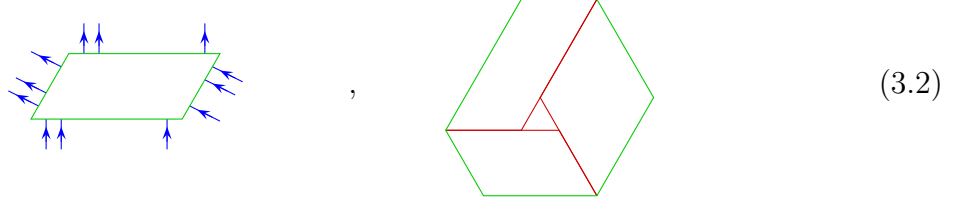
$$180(n - 2) = 300(n - 6) + \sum_i^3 \alpha_i + \beta_i.$$

And we can simplify it to have $1440 - 120n = \sum \alpha_i + \beta_i$. Since β_i is either 120 or 240, n can be either 6, 7 or 8. For each n , we look at the all possible combinations of α_i, β_j up to symmetries. Then we check whether it actually bounds a polygon. For example, $n = 6$ there are 6 possible combinations of angles but one does not bound a polygon, $(\alpha_i) = (0, 120, 240), (\beta_i) = (120, 120, 120)$. For $n > 6$, one has to use $n - 6$ times of 300 angles. If some of a_i, b_j are zero then we can play the same game to find all shapes, we might have 60 for some β_j . The polygon in the middle might be a triangle, a rectangle or a pentagon but we will consider them as a special case of a hexagon. But we can obtain all these possible shapes from the general shape by substituting some zeros. Then we prove the following lemma. Let $d = \text{Min} \{a_1, a_2, a_3, b_1, b_2, b_3\}$.

Lemma 3.3 $\dim(\text{Inv}(V_{a_1\lambda_1+b_1\lambda_2} \otimes V_{a_2\lambda_1+b_2\lambda_2} \otimes V_{a_3\lambda_1+b_3\lambda_2}))$ is $d + 1$ if there exist non negative integers $i, j, k, l, m, n, o, p, q$ such that $a_1 = i + p, a_2 = j + n, a_3 = k + l, b_1 = j + o, b_2 = k + m, b_3 = i + q$ and $0 = le^{\frac{\pi}{3}i} + me^{\frac{\pi}{6}i} + n + oe^{-\frac{\pi}{6}i} + pe^{-\frac{\pi}{3}i} - q$. Otherwise, it is zero.

Proof: One might directly find the answer by using the decomposition of the tensor product $V_{a_1\lambda_1+b_1\lambda_2} \otimes V_{a_2\lambda_1+b_2\lambda_2} \otimes V_{a_3\lambda_1+b_3\lambda_2}$ into irreducible representations. But one can see that these are conditions we can easily obtain from the figure (3.1) and the last one is one condition that the hexagon in the middle does exist. \square

Since the parallelogram in figure (3.2) changes the directions of edges (the interior is filled by the unique maximal hexagonal cut out), we can push the hexagon to an equilateral triangle (possibly empty). The size of this equilateral triangle is the minimum of differences of lengths of three pairs of parallel edges of the hexagon.



But the resulting shape has mixed aspects. First it might contain some non-segregated clasps which did not exist before. When we apply single or double clasp expansions to these non segregated clasps, it is possible to get one by transforming the basis by adding H 's but usually it becomes very difficult to deal with. If we keep the hexagonal shape, it usually produces multiple non-vanishing terms in a single clasp expansion. So we will use both shapes depend on the feasibility.

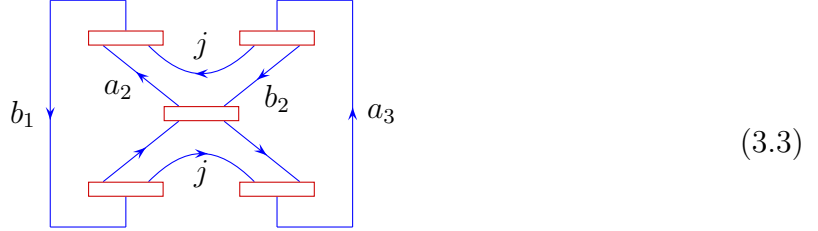
3.2 Proofs of Theorems

Since $M_\Theta(\lambda)$ is an 1×1 matrix, we will write it as a scalar through this section. For nontrivial planar Θ shapes with $d = 0$, we could have at most three zeros for a_i, b_j . Before we prove the main theorems we start with cases of two zeros.

3.2.1 Two Zeros

If we have two zeros, up to symmetries, it is either one of these subcases: 1) $a_1 = b_3 = 0$, 2) $a_1 = a_3 = 0$.

If $a_1 = b_3 = 0$, there exists j such that $b_1 = a_2 + j$, $a_3 = b_2 + j$ and its shape is



By the same idea of the proof of proposition 2.14, we can easily get the following proposition 3.4. Let $i = a_2$ and $k = b_2$.

Proposition 3.4 $M_\Theta(0, i + j, i, k, j + k, 0)$ is

$$\frac{[i + j + k + 2]![i + 1]![j]![k + 1]!}{[i + j]![j + k]![i + k + 1]![2]}.$$

Proof: Let us use the notation $[i, j, k]$ for the trihedron coefficient of this Θ shape.

We induct on k . If $k = 0$,

$$[i, j, 0] = \frac{[i + j + 2][i + j + 1]}{[2]}.$$

For $k \neq 0$, by the idea of the proof of proposition 2.14 and the induction hypothesis, we have

$$\begin{aligned} [i, j, k] &= \frac{[j + k + 2]}{[j + k]}[i, j, k - 1] - \frac{[i + 1][i]}{[i + k + 1][i + k]}[i - 1, j + 1, k - 1] \\ &= \frac{[i + j + k + 2]![i + 1]![j]![k + 1]!}{[i + j]![i + k + 1]![j + k]!} \left(\frac{[j + k + 2][i + k + 1] - [j + 1][i]}{[i + j + k + 2][k + 1]} \right) \\ &= \frac{[i + j + k + 2]![i + 1]![j]![k + 1]!}{[i + j]![i + k + 1]![j + k]!} \end{aligned}$$

because $[j + 1 + k + 1][i + k + 1] = [j + 1][i] + [i + j + k + 2][k + 1]$. □

If $a_1 = a_3 = 0$, there exists k such that $b_1 = b_2 + k$, $a_2 = b_3 - b_2 + k$ and its shape is

(3.4)

We need to prove a sequence of lemmas. First, we prove that the clasp in the middle is not essential.

Lemma 3.5 *Let $n \geq 1$, then*

(3.5)

Proof: The idea of the proof is that if we have any Y 's in the single clasp expansion at the middle clasp, it becomes zero. The argument, we used to find the general shape, leads us that there does not exist a filling with boundary $(0, n, 0, n, 1, n - 2)$. Thus, it has to vanish once we have any Y 's. \square

Form the third figure in equation (3.5), we apply a single clasp expansion to the clasp in the left. For the following equation, it should be clear without the direction of edges.

Lemma 3.6 *Let $n \geq 1$, then*

$$\begin{array}{c} \text{Diagram 1} \end{array} = (-1)^n [n+1] \begin{array}{c} \text{Diagram 2} \end{array} \quad (3.6)$$

Diagram 1: A diagram with two green triangles, one pointing up and one pointing down, connected by a vertical blue arrow labeled n . A blue arc labeled n connects the left side of the top triangle to the left side of the bottom triangle. Each triangle has a red rectangle on its right side.

Diagram 2: A diagram with two red rectangles, one above the other, connected by a blue arc labeled n .

Proof: We induct on n . If $n = 1$, the coefficient is $-[2] = (-1)^1[2]$. We apply a single clasp expansion at the left clasp which gives us the first equality in the following equation (3.8).

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} + \frac{[n-1]}{[n]} \begin{array}{c} \text{Diagram 3} \end{array} \quad (3.7)$$

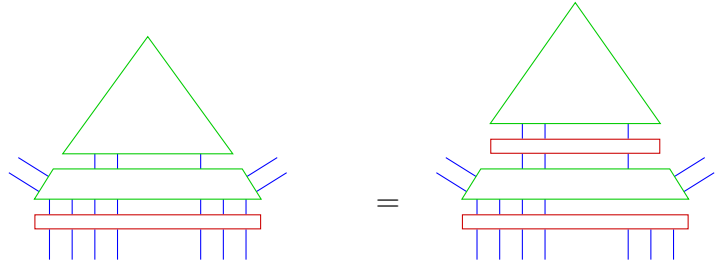
$$= \begin{array}{c} \text{Diagram 4} \end{array} + \frac{[n-1]}{[n]} \begin{array}{c} \text{Diagram 5} \end{array} + \frac{[n-2]}{[n]} \begin{array}{c} \text{Diagram 6} \end{array}$$

$$= \alpha \begin{array}{c} \text{Diagram 7} \end{array} + \beta \frac{[n-1]}{[n]} \begin{array}{c} \text{Diagram 8} \end{array} + \frac{[n-2]}{[n]} \begin{array}{c} \text{Diagram 9} \end{array}$$

The diagrams in (3.7) and (3.8) are more complex versions of the ones in (3.6), involving multiple clasp expansions and additional blue arcs and rectangles. Diagram 4 has a blue arc labeled $n-1$ and a blue arc labeled 1 . Diagram 5 has a blue arc labeled $n-2$ and a blue arc labeled 2 . Diagram 6 has a blue arc labeled $n-2$ and a blue arc labeled 2 . Diagram 7 has a blue arc labeled $n-1$ and a blue arc labeled 1 . Diagram 8 has a blue arc labeled $n-2$ and a blue arc labeled 2 . Diagram 9 has a blue arc labeled $n-2$ and a blue arc labeled 2 .

where $\alpha = (-1)^{n-1}[n]$ and $\beta = (-1)^{n-2}[n-1]$.

For the second equality, we apply a single clasp expansion for the resulting clasp of weight $(n-1)\lambda_2$ for which clasps of weight $(n-2)\lambda_2$ are located at the northeast corner. It is not difficult to see the following equality. The number of strings coming from the trapezoid can be any integer between 0 and n where n is the weight of the clasp given the left side of equality.



Thus, we can put a clasp of weight $(n-1)\lambda_2((n-2)\lambda_2, (n-2)\lambda_2)$ at the gap between an equilateral triangle and a trapezoid at the first (second and third, respectively) figure in the second line of equation (3.8). Therefore, we can use induction to get the third equality in equation (3.8). Note that the size of the third equilateral triangle in the third line is $n-2$.

Last step is to count how many $-[2]$'s will be produced when we change it to multiple of the web in the right hand side of equation (3.6). But it is fairly easy to see that each of them has just one factor of $-[2]$ in the first two in the third line. If we add up the all coefficients, we have

$$\begin{aligned}
 & -[2](-1)^{n-1}[n] - [2](-1)^{n-2} \frac{[n-1][n-1]}{[n]} + (-1)^{n-2} \frac{[n-1][n-2]}{[n]} \\
 & = (-1)^n \frac{1}{[n]} ([2][n]^2 - [2][n-1]^2 + [n-2][n-1]) \\
 & = (-1)^n \frac{1}{[n]} [n+1][n] = (-1)^n [n+1]
 \end{aligned}$$

□

The lemma 3.6 can be generalized to the following lemma.

Lemma 3.7

$$= (-1)^j \frac{[i+j+1]}{[i+1]} \quad (3.8)$$

Proof: We induct on $i+j$. If $i=1, j=0$, the coefficient is $1 = (-1)^1 \frac{[2]}{[2]}$. If $i=0$, it follows from the previous lemma. The first equality can be proven with a single clasp expansion at the left middle clasp in left figure. The second equality can be proven the same argument we use in lemma. The rest of proof follows by induction. \square

Lemma 3.8

$$= (-1)^{j+1} \frac{[i+j+k+1]![i]![j]![k]!}{[i+j]![j+k]![i+k+1]!} \quad (3.9)$$

Proof: We induct on k . If $k=0$, it follows from the previous lemma. If $k \neq 0$, we use a single clasp expansion at the left middle clasp. Then we get the following equality.

By induction, the coefficient is equal to

$$\begin{aligned}
& (-1)^j \frac{[i+j+k]![i]![j]![k-1]!}{[i+j]![i+k]![j+k-1]!} \\
& + \frac{[j][j]}{[j+k][j+k-1]} (-1)^{j-1} \frac{[i+j+k]![i+1]![j-1]![k-1]!}{[i+j]![i+j+1]![j+k-2]!} \\
& = (-1)^j \frac{[i+j+k+1]![i]![j]![k]!}{[i+j]![j+k]![i+k+1]!} \left(\frac{[i+k+1][j+k] - [i+1][j]}{[i+j+k+1][k]} \right) \\
& = (-1)^j \frac{[i+j+k+1]![i]![j]![k]!}{[i+j]![j+k]![i+k+1]!}
\end{aligned}$$

because $[i+1+k][j+k] = [i+1][j] + [i+j+k+1][k]$. □

Proposition 3.9 $M_{\Theta}(0, b_1, a_2, b_2, 0, b_3)$ is

$$(-1)^{b_2+1} \frac{[b_1 - b_2 + b_3 + 2]![b_1 - b_2]![b_3 - b_2]![b_2 + 1]!}{[b_1]![b_1 - 2b_2 + b_3]![b_3]![2]}.$$

or $M_{\Theta}(0, i+j, i+k, j, 0, j+k)$ is

$$(-1)^j \frac{[i+j+k+2]![i]![j+1]![k]!}{[i+j]![j+k]![i+k]![2]}.$$

Corollary 3.10 $M_{\Theta}(0, n, 0, n, 0, n)$ is

$$(-1)^n \frac{[n+1]^2[n+2]}{[2]}.$$

3.2.2 Proof of Theorem 3.1

We start to prove the following lemma to find the trihedron coefficient.

Lemma 3.11

(3.10)

Proof: We use an expansion in equation (2.24) at the right middle clasp. Then there are $\min\{l, m + j\}$ terms in the expansion. But once we have a U turn, we will show that it becomes zero. If there is a U turn we use a single clasp expansion at the top-left clasp of weight $m + j$. Then all terms vanish except one term which has Y joining the top right clasp of weight $(i + l, j + m)$ and the triangle in the top center. Then, there is a sequence of H 's we can push that move the entire shape by one string. Eventually Y has to join two strings from the triangle but we knew from lemma 3.12 that it becomes zero. Therefore, we can free l strings from the right middle clasp of weight $(l, j + m)$. \square

Unfortunately this lemma 3.11 is not true if $k \neq 0$. Actually only two terms survive but there is a layer of H 's which makes the problem difficult in this approach. Continuing the proof of the theorem, lemma 3.11 implies that $M_{\Theta}(0, i + j, l, j + m, i + m, j + l)$ is

$$\frac{[j + l + 1][i + j + l + m + 2]}{[j + 1][i + j + m + 2]} M_{\Theta}(0, i + j, 0, j + m, i + m, j).$$

Then the result follows by proposition 3.9.

3.2.3 Proof of Theorem 3.2

We use the same idea of lemma 3.5 but for $j = 0$, all terms in this expansion do not vanish. For next step we need to show the following lemma.

Lemma 3.12 *Let $\alpha = 0$ if $n > \min\{l, k\}$, $(\frac{[k]^n [l+k-n]!}{[l+k]!})^2$ if $n \leq \min\{l, k\}$. Then*

The diagram shows an equality between two web configurations. The left web has four red rectangular clasp nodes. Blue strands enter from the top and exit from the bottom. Labels include i, l, m at the top right, k, n near the top clasp nodes, and $m-n$ near the bottom clasp nodes. The right web is simpler, with two red rectangular clasp nodes. Labels include $i+n$ at the top right, m near the top clasp node, and $k-n$ near the bottom clasp node. The equation is labeled (3.11) on the right.

Proof: The clasp of weight $m\lambda_2$ can be pushed into the clasp of weight $l\lambda_1 + (i+m)\lambda_2$. For the clasp of weight $(k+n)\lambda_1$, we use a single clasp expansion. \square

First we use the equation (2.24) at the middle clasp of weight $(k+l)\lambda_1 + m\lambda_2$. By lemma 3.12 we can transform each web to the web in the righthand side of lemma 3.12.

The diagram shows an equality between two trihedron shapes. The left shape has three vertical red bars. The top bar has three horizontal blue lines extending to the left, labeled i , l , and m from top to bottom. The middle bar has a single horizontal blue line extending to the left, labeled k . The bottom bar has three horizontal blue lines extending to the left. The right shape is a sum over n from 0 to $\min\{l+k, m\}$ of a_n times a trihedron shape. This shape has four vertical red bars. The top bar has three horizontal blue lines extending to the right, labeled i , l , and m from top to bottom. The second bar from the left has a horizontal blue line extending to the right, labeled k . The third bar from the left has a horizontal blue line extending to the right, labeled n . The bottom bar has three horizontal blue lines extending to the right, with the middle one labeled $m-n$. The label (3.12) is at the end of the equation.

Since its shape can be written as $[k-n, i+n, m]$, by proposition 3.4 it has value

$$\propto \frac{[i+k+m+2]![k-n+1]![i+m]![m+1]!}{[i+k]![i+m+n]![k+m-n+1]![2]}.$$

Therefore, it completes the proof.

Chapter 4

A Complete Set of Relations of $\mathcal{U}_q(\mathfrak{sl}(4, \mathbb{C}))$

Our webs are generated by the two shapes of trivalent vertices.



And the following is our conjecture for a complete set of relations for $\mathcal{U}_q(\mathfrak{sl}(4, \mathbb{C}))$.

$$\begin{array}{c} \text{circle with arrow} \end{array} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad (4.1)$$

$$\begin{array}{c} \text{double circle} \end{array} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad (4.2)$$

$$\begin{array}{c} \text{double line with loop} \end{array} = [2] \begin{array}{c} \text{double line} \end{array} \quad (4.3)$$

$$\begin{array}{c} \text{line with loop} \end{array} = [3] \begin{array}{c} \text{line} \end{array} \quad (4.4)$$

$$\begin{array}{c} \text{Y-junction with double line} \end{array} = \begin{array}{c} \text{X-junction with double line} \end{array} \quad (4.5)$$

$$\begin{array}{c} \text{Y-junction with double line} \end{array} = \begin{array}{c} \text{X-junction with double line} \end{array} \quad (4.6)$$

$$\begin{array}{c} \text{square with double lines} \end{array} = [2] \begin{array}{c} \text{two arcs} \end{array} + \begin{array}{c} \text{two arcs} \end{array} \quad (4.7)$$

$$\begin{array}{c} \text{square with double lines} \end{array} = \begin{array}{c} \text{Y-junction with double line} \end{array} + \begin{array}{c} \text{two arcs} \end{array} \quad (4.8)$$

$$\begin{array}{c} \text{square with double lines} \end{array} = \begin{array}{c} \text{square with double lines} \end{array} \quad (4.9)$$

$$\begin{array}{c} \text{hexagon with double lines} \end{array} = \begin{array}{c} \text{hexagon with double lines} \end{array} - \begin{array}{c} \text{two arcs} \end{array} + \begin{array}{c} \text{two arcs} \end{array} \quad (4.10)$$

First of all, we compute the dimension of invariant space of all tensor products of 4, 6 fundamental representation of $\mathcal{U}_q(\mathfrak{sl}(4, \mathbb{C}))$. There is a general way to find a basis webs with a fixed boundary, all of them are fundamental representation, for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ and $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$ [KK99]. It is still unknown how we can actually find all basis webs of all possible boundaries of fundamental representations. But for our case, there are only few boundaries and we can find the dimension and even find a basis webs without difficulty, most of basis webs do not have any faces.

Since all webs are generated by two trivalent vertices, by multiplying a complex number, we can have a different set of generators. Therefore, we have two choices of freedom to set any two independent coefficients. Let a, b, c, d, e and f be unknowns in equation 4.4, 4.6, 4.7 and 4.8. By the quantum Weyl formula, we do know the value of the first two equations. We use the first choice of freedom to have the equation 4.3. The following equality implies $a = [3]$.

$$\begin{aligned}
 \text{Diagram 1} &= a \text{Diagram 2} = [4]a \\
 &= [2] \text{Diagram 3} = [2] \frac{[4][3]}{[2]} = [4][3]
 \end{aligned}$$

We use the last choice of freedom to have equation 4.5. To get the equation 4.6 (which is actually the dual of the equation 4.5), we start from the following equations.

$$\text{Diagram 1} = \alpha \text{Diagram 2} + \beta \text{Diagram 3}$$

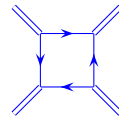
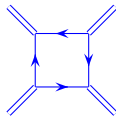

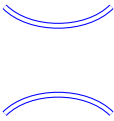
Using 4.5 and 4.6 (with an unknown variable a), we found $\alpha = 0$ and $\beta = [2]b$. By attaching a U turn on the top of each webs in the equation, we get $[2][3] = [4]\alpha + [3]\beta$. Thus $b = 1$. For equation 4.7, we attach U turns on the top and right side of

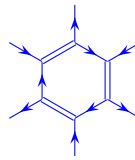
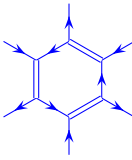

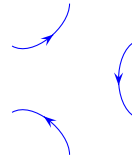
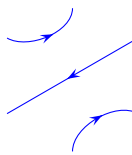

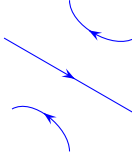
each webs in the equation. Then the resulting web can be expanded as a linear combination of a basis webs of different boundary. By comparing the coefficients, we get $[3][3] = [4]c + d$, $[2][3] = c + [4]d$. It is easy to find that $c = [2]$, $d = 1$. For equation 4.8 we attach


(4.11)

to right side of each webs to get $e = f = 1$.

For last two equations 4.9 and 4.10, we need to start from the following equations.


 $= g$

 $+ h$

 $+ i$

(4.12)


 $= j$

 $+ k$

 $+ l$

 $+ m$

 $+ n$

 $+ p$

(4.13)

By attaching U turns and H (as in 4.11) for equation 4.9, we get

$$[2][3] = [2][3]g + h + \frac{[4][3]}{[2]}i$$

$$[2][3] = [2][3]g + \frac{[4][3]}{[2]}h + i$$

$$[2] = [2]g + [3]h$$

$$[2] = [2]g + i$$

One can solve them to have $g = 1$ and $h = i = 0$. For the equation 4.10, we just need to attach H (as in 4.11) to right top side of each basis webs in the equation 4.13. Then we follow the same procedure to get the following six equations: $j = 1$, $[2]^2 j + [3]k = 1$, $[2]j + [3]p = [2]$, $l = 1$, $m = 0$ and $n = 0$.

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